# Appendices

## Appendix 1. Construction of the indirect-damage rate function

Pumping groundwater from aquifer systems leads to compaction of compressible fine-grained sediments within or next to aquifer systems (Leake and Prudic, 1991, p.1). In an opposite manner, aquifers with coarse-grained sediments compaction may be reversed when groundwater is replenished, given the fact that they have no impact on the aquifer system's storage capacity (Williamson et al., 1989, p.97). All aquifer systems, according to Holzer and Galloway (2005), compact to some degree in response to a change in groundwater level. Compaction is controlled, in theory, by effective stress (Holzer and Galloway, 2005). As suggested by Terzaghi (1925), effective stress (total pressure (geostatic stress) minus the pore-fluid pressure (neutral or hydrostatic stress)) is given by Equation (1.1) below

$$\delta' = \delta - \mu \tag{1.1}$$

where $\delta'$, $\delta$, and $\mu$ represent the effective stress, the total pressure (geostatic stress), and the pore-fluid pressure (neutral or hydrostatic stress), respectively. Removing groundwater from sediments lowers the pore-fluid pressure within the sediments (Holzer and Galloway, 2005). As a result, the effective stress rises, and the pore space (or pore volume) decreases. This process is referred to by hydrologists as compaction (Poland et al., 1972). The change in effective stress has been shown to be proportional to the amount of compaction (Riley, 1969; Helm 1975; Leake and Prudic, 1991, p.3). The change in effective stress in an unconfined aquifer system depends on the change in water table level (Leake and Prudic, 1991, p.3). Thus, we define the change in effective stress for an unconfined aquifer system as suggested by Poland and Davis (1969, p.195)

$$\Delta\delta' = -\gamma_w(1 - n + n_w)\Delta H \tag{1.2}$$

where $\Delta\delta'$ represent the change in effective stress (positive for a rise and negative for a reduction), , $\gamma_w$ represents the unit weight of water ($N/m^3$), $n$ represents the porosity (dimensionless), $n_w$ represents the moisture content of sediments above water table (in the unsaturated zone) as a fraction of total volume (dimensionless), and $\Delta H$ represents the change in water table (positive for raising and negative for lowering). The change in water table height per unit time is give by (Koundouri, 2004; Latinopoulos and Sartzetakis, 2014)

$$\dot{H} = \frac{1}{AS}[R - (1-\alpha)W]. \tag{1.3}$$

In Equation (1.2), we replace $\Delta H$ with $\frac{1}{AS}[R - (1-\alpha)W]$ such that Equation (1.2) becomes

$$\Delta\delta' = -\gamma_w(1 - n + n_w)\frac{1}{AS}[R - (1-\alpha)W]. \tag{1.4}$$

Therefore, $\Delta H = \frac{1}{AS}[R - (1-\alpha)W]$ can be negative (implying the water table height is lowering) if $\mathrm{W} > \mathrm{R} + \alpha W$ and this represents groundwater drawdown which contributes to LS. When $\mathrm{W} < \mathrm{R} + \alpha W$, $\Delta H$ is positive which implies a reduction in effective stress and thus no compaction (or LS) occurrence. In this case, farmers will not be taxed since they have been preventing LS from occuring (Wang et al., 2015). That is, we assume that the pumpers are penalized for any of their action (in this case, simply withdrawals) that leads to inelastic compaction (a permanent reduction in the thickness of sediments due to an increase in vertical effective stress).

Compaction, on the other hand, occurs whenever there is an increase in the effective stress. However, inelastic (permanent) compaction, which results in the loss of aquifer system storage capacity, occurs only when the effective stress exceeds the pre-consolidation stress (Holzer and Galloway, 2005; Lofgren, 1975, p.40). Pre-consolidation stress refers to the highest effective stress that a soil has experienced over its life (Yang et al., 2009). Any rise in effective stress value lower than the pre-consolidation stress causes elastic compaction, in which sediment deformations can be reversed by replenishing the aquifer system. When inelastic compaction occurs, pore space is permanently reduced and cannot be restored. This means that the aquifer system's storage capacity is reduced forever. Even if the aquifer system's water level is restored throughout, it will not be able to contain the same volume of water as it did before the compaction (Williamson et al., 1989, p.97). When the aquifer system experiencing subsidence is replenished and then groundwater levels fall again, significant compaction will not resume until the new pre-consolidation stress is surpassed (Holzer and Galloway, 2005; Leake and Prudic, 1991, p.4). As suggested by Poland (1969, p.288-290), the approximate inelastic compaction $\Delta q$ (in $m$) is given by Equation (1.5) below

$$\Delta q = y_v y \Delta\delta' \tag{1.5}$$

where $y_v$ and $y$ represent the compacting beds' mean compressibility and the aggregate thickness, respectively. As mentioned earlier, we assume a single-cell aquifer system with a non-heterogeneous distribution of impacts and wells. Without loss of generality, we assume that the compacting beds' aggregate thickness is equal to the aquifer system's thickness $b$, and that the compacting beds' mean compressibility is equal to the aquifer system compressibility $\psi$. The aquifer's system storage capacity is represented by $AS$. The total amount of water resource presented in the aquifer system at a given time is obtained by multiplying $H$ by $AS$ (Williams et al., 2017). The entire amount of storage capacity lost as a result of inelastic compaction is therefore obtained by multiplying $AS$ by $\Delta q$ ($\Delta\overline{q} = AS\psi b\Delta\delta'$).

To determine the level of tax ($\phi$) levied on farmers for contributing to the reduction of aquifer system storage capacity, we must first derive the shadow price of aquifer system storage capacity. We know that the shadow price for aquifer system storage capacity represents the opportunity cost of losing aquifer system storage capacity, which is what farmers had to give up when they chose to extract water excessively, reducing aquifer system storage capacity. The storage capacity of the aquifer system and the extractions are both measured in volume (cubic meters) in this study.

Therefore, the shadow price per cubic meter of aquifer system storage capacity equals the net income of farmers per cubic meter of irrigation water that can be stored in the aquifer at that unit cubic meter space. As a result, we observe that the net income (total revenue minus total cost) of farmers is given by $\frac{W^2}{2k} - \frac{gW}{k} - (C_0 + C_1H)W$. To obtain the marginal net income of farmers (the increase in net income due to extracting and consuming one additional cubic meter of irrigation water), we differentiate $\frac{W^2}{2k} - \frac{gW}{k} - (C_0 + C_1H)W$ with respect to W and obtain $\frac{W}{k} - \frac{g}{k} - (C_0 + C_1H)$. In other words, we observe that farmers obtain a net income in the amount $\frac{W}{k} - \frac{g}{k} - (C_0 + C_1H)$ per additional cubic meter of irrigation water extracted. As a result, the tax ($\phi$) is equal to the farmers' marginal net income per cubic meter of irrigation water extracted. Since $\frac{W}{k} - \frac{g}{k} - (C_0 + C_1H)$ is not fixed but instead varies depending on $W$ and $H$, the Pigouvian tax $\phi$ is not a fixed tax but a proportional tax. In terms of aquifer system storage capacity, $\phi$, is given by $\phi(W,H) = \frac{W}{k} - \frac{g}{k} - (C_0 + C_1H)$ per cubic meter of aquifer system storage capacity lost.

Therefore, the shadow price of aquifer system storage capacity lost due to groundwater extraction is given by $\phi(W, H)\Delta\bar{q}$.

**Appendix 2. Proof of sub-problem 1.**

The hamiltonian function of the system (9), (10), (11) is given as follows

$$\mathcal{H}_2(t, W_2, H_2, \lambda_2) = -\mathrm{e}^{-it}[\frac{W_2^2}{2k} - \frac{gW_2}{k} - (C_0 + C_1H_2)W_2 + \frac{\beta \cdot \eta \cdot \varepsilon \cdot b \cdot \psi}{AS}$$

$$[R - (1-\alpha)W_2] + b\psi\gamma_w(1 - n + n_w)[R - (1-\alpha)W_2]$$

$$(\frac{W_2}{k} - \frac{g}{k} - C_0 - C_1H_2)] + \lambda_2 \cdot \frac{[R + (\alpha - 1)W_2]}{\Omega \cdot AS} \quad (2.1)$$

Equation (2.1) can be rewritten as follows

$$\mathcal{H}_2(t, W_2, H_2, \lambda_2) = -\mathrm{e}^{-it}[\frac{W_2^2}{2k} - \frac{gW_2}{k} - (C_0 + C_1H_2)W_2 + G_5W_2$$

$$-G_3\frac{(1-\alpha)W_2^2}{k} - G_3RC_1H_2 + G_3(1-\alpha)C_1W_2H_2$$

$$+G4] + \lambda_2 \cdot \frac{[R + (\alpha - 1)W_2]}{\Omega \cdot AS} \quad (2.2)$$

Where

$$G_2 = \frac{\beta \cdot \eta \cdot \varepsilon \cdot b \cdot \psi}{AS}. \quad (2.3)$$

$$G_3 = b\psi\gamma_w(1 - n + n_w). \quad (2.4)$$

$$G_4 = -\frac{RgG_3}{k} - RC_0G_3 + G_2R. \quad (2.5)$$

$$G_5 = \frac{RG_3}{k} + \frac{(1-\alpha)gG_3}{k} + G_3(1-\alpha)C_0 - G_2(1-\alpha). \quad (2.6)$$

Hence, the first order conditions are as follows

$$\frac{\partial \mathcal{H}_2}{\partial W_2} = -\mathrm{e}^{-it}[(\frac{1-2G_3(1-\alpha)}{k})W_2 - \frac{g}{k} - C_0 - C_1H_2 + G_5 + G_3(1-\alpha)C_1H_2] + \lambda_2[\frac{(\alpha-1)}{\Omega\cdot AS}] = 0. \quad (2.7)$$

$$\dot{\lambda}_2 = -\frac{\partial \mathcal{H}_2}{\partial H_2}. \quad (2.8)$$

$$\dot{H}_2 = \frac{1}{\Omega\cdot AS}[R + (\alpha-1)W_2]. \quad (2.9)$$

The transversality condition is given by $\lim_{t\to\infty}\lambda_2(t) = 0$. From Equation (2.7), we obtain the value for the costate variable $\lambda_2$ as follows.

$$\lambda_2 = \frac{\Omega}{m}\mathrm{e}^{-it}[(\frac{1-2G_3(1-\alpha)}{k})W_2 - \frac{g}{k} - C_0 - C_1H_2 + G_5 + G_3(1-\alpha)C_1H_2], \quad (2.10)$$

where $m = \frac{(\alpha-1)}{AS}$. The derivative of $\lambda_2$ with respect to $t$ is given by

$$\dot{\lambda}_2 = \frac{\Omega}{m}\mathrm{e}^{-it}[-\frac{iG_7W_2}{k} + \frac{ig}{k} + iC_0 - G_6iC_1H_2 - iG_5 + \frac{G_6C_1R}{\Omega\cdot AS} + G_6C_1\frac{m}{\Omega}W_2 + \frac{\dot{G}_7W_2}{k}]. \quad (2.11)$$

Where

$$G_6 = G_3(1-\alpha) - 1. \quad (2.12)$$

$$G_7 = 1 - 2G_3(1-\alpha). \quad (2.13)$$

The derivative of $\mathcal{H}_2$ with respect to the water table elevation $H_2$ is given by

$$-\frac{\partial \mathcal{H}_2}{\partial H_2} = -\mathrm{e}^{-it}[G_3RC_1 - G_6C_1W_2]. \quad (2.14)$$

From Equation (2.8) and (2.11), we obtain the following equation.

$$-G_3RC_1 + G_6C_1W_2 = \frac{\Omega}{m}\mathrm{e}^{-it}[-\frac{iG_7W_2}{k} + \frac{ig}{k} + iC_0 - G_6iC_1H_2$$

$$-iG_5 + \frac{G_6C_1R}{\Omega\cdot AS} + G_6C_1\frac{m}{\Omega}W_2 + \frac{\dot{G}_7W_2}{k}]. \quad (2.15)$$

Solving for $\dot{W}_2$ in the above equation we get the following equations.

$$\frac{\Omega \cdot G_7 \dot{W}_2}{mk} = \frac{\Omega \cdot i G_7 W_2}{mk} + \frac{\Omega \cdot i C_1 G_6 H_2}{m} - \frac{\Omega \cdot ig}{mk} - \frac{\Omega \cdot i C_0}{m} + \frac{\Omega \cdot i G_5}{m} - \frac{G_6 C_1 R}{m \cdot AS} - G_3 R C_1 \quad (2.16)$$

$$\frac{G_7 \dot{W}_2}{k} = \frac{i G_7 W_2}{k} + i C_1 G_6 H_2 - \frac{ig}{k} - i C_0 + i G_5 - \frac{G_6 C_1 R}{\Omega \cdot AS} - \frac{m}{\Omega} G_3 R C_1 \quad (2.17)$$

$$\dot{W}_2 = i W_2 + \frac{ik C_1 G_6 H_2}{G_7} - \frac{ig}{G_7} - \frac{ik C_0}{G_7} + \frac{ik G_5}{G_7} - \frac{k G_6 C_1 R}{\Omega \cdot AS G_7} - \frac{mk}{\Omega G_7} G_3 R C_1 \quad (2.18)$$

$$\dot{W}_2 = i W_2 + \frac{ik C_1 G_6 H_2}{G_7} + [-\frac{ig}{G_7} - \frac{ik C_0}{G_7} + \frac{ik G_5}{G_7} - \frac{k G_6 C_1 R}{\Omega \cdot AS G_7} - \frac{mk}{\Omega G_7} G_3 R C_1]. \quad (2.19)$$

Likewise, the value for $\dot{H}_2$ can be rewritten as

$$\dot{H}_2 = \frac{(\alpha-1) W_2}{\Omega \cdot AS} + \frac{R}{\Omega \cdot AS}. \quad (2.20)$$

Consequently, we now have to solve the two simultaneous differential equations ((2.19) and (2.20)). Thus, by letting $mm = \frac{(\alpha-1)}{\Omega \cdot AS}$, $uu = ikC_1 \frac{G_6}{G_7}$, $NN = \frac{1}{G_7}[-ig - ikC_0 + ikG_5 - \frac{kG_6 C_1 R}{\Omega \cdot AS} - \frac{mk}{\Omega} G_3 R C_1]$ and $MM = \frac{R}{\Omega \cdot AS}$, we get the following system of differential equations.

$$\dot{W}_2 = i W_2 + uu \cdot H_2 + NN. \quad (2.21)$$

$$\dot{H}_2 = mm \cdot W_2 + MM. \quad (2.22)$$

Putting the above system of differential equations in a $D$ operator format (where $D = \frac{d}{dt}$), and solving for $W_2$ yields the following second order linear non-homogeneous differential equation.

$$[(D^2 - Di) - uu \cdot mm] W_2 = uu \cdot MM. \quad (2.23)$$

The particular solution of the above differential equation is given by: $-\frac{MM}{mm}$ and the solution to the homogeneous differential equation ($[(D^2 - Di) - uu \cdot mm] W_2 = 0$) by

$$W_2(t) = \overline{EA} \mathrm{e}^{t x_1} + \overline{EB} \mathrm{e}^{t x_2}, \quad (2.24)$$

where $x_{1,2} = \frac{i \pm \sqrt{i^2 + 4uumm}}{2}$ are the characteristic roots. The parameters $\overline{EA}$ and $\overline{EB}$ are constants to be determined by imposing the initial conditions. Substituting the right hand side (RHS) of (2.24) for $W(t)$ in the homogenous DE ($\dot{H}_2 = mm \cdot W_2$) and integrating gives the solution for the water table level $H(t)$ as follows.

$$H_2(t) = \frac{mm \cdot \overline{EA}}{x_1} \mathrm{e}^{tx_1} + \frac{mm \cdot \overline{EB}}{x_2} \mathrm{e}^{tx_2}. \quad (2.25)$$

Furthermore, the steady state level water table is given by

$$H_2^* = [\frac{i\frac{MM}{mm} - NN}{uu}] \quad (2.26)$$

Hence, the solution for $W_2^*(t)$ and $W_2^*(t)$ are given as follows, respectively.

$$W_2(t) = \overline{EA}\mathrm{e}^{tx_1} + \overline{EB}\mathrm{e}^{tx_2} - \frac{MM}{mm}, \quad (2.27)$$

$$H_2(t) = \frac{mm \cdot \overline{EA}}{x_1} \mathrm{e}^{tx_1} + \frac{mm \cdot \overline{EB}}{x_2} \mathrm{e}^{tx_2} + \frac{i\frac{MM}{mm} - NN}{uu}. \quad (2.28)$$

Similarly to Gisser and Sanchez (1980) results, it is worth mentioning that $+4uumm > 0$ since $k < 0, C_1 < 0, i > 0, A > 0, S > 0, \Omega > 0, \psi > 0, \gamma_w > 0, b > 0, n > 0, n_w > 0, G_3 > 0, G_6 < 0, G_7 > 0$, and $\alpha < 1 \Longrightarrow (\alpha - 1) < 0$ or $(1 - \alpha) > 0$. This implies that $x_1 > i$ and $x_2 < 0$. This implies that $x_1 > i$ and $x_2 < 0$. Therefore, $x_2$ is the stable characteristic root. Likewise, similarly to Gisser and Sanchez (1980), we obtained that the transversality condition is only satisfied when $\overline{EA} = 0$. By imposing the initial conditions of the sub problem ($H_2(t_T) = H_T$), we obtain the constant $\overline{EB}$ as follows below.

$$\overline{EB} = \frac{x_2}{mm}[H_T - \frac{i\frac{MM}{mm} - NN}{uu}]\mathrm{e}^{-x_2 t_T}. \quad (2.29)$$

Therefore, the optimal solutions for $W_2^*(t)$ and $H_2^*(t)$ are given as follows below, respectively.

$$W_2^*(t) = \frac{x_2}{mm}[H_T - \frac{i\frac{MM}{mm} - NN}{uu}]\mathrm{e}^{x_2(t - t_T)} - \frac{MM}{mm}. \quad (2.30)$$

$$H_2^*(t) = [H_T - \frac{i\frac{MM}{mm} - NN}{uu}]\mathrm{e}^{x_2(t - t_T)} + \frac{i\frac{MM}{mm} - NN}{uu}. \quad (2.31)$$

Because $x_2 < 0$ and $i > 0$, the functional defined in (9) is verified to be a convergent integral.

**Appendix 3. Proof of sub-problem 2**

We can now solve the first sub-problem since we have the solution ($SP_2^*$) to the second sub problem. The hamiltonian function of the system (16), (17), (18) is given as follows

$$\mathcal{H}_1(t, W_1, H_1, \lambda_2) = -\mathrm{e}^{-it}[\frac{W_1^2}{2k} - \frac{gW_1}{k} - (C_0 + C_1H_1)W_1 + G_2[R - (1-\alpha)W_1]]$$
$$+\lambda_1 \cdot \frac{[R+(\alpha-1)W_1]}{AS} \quad (3.1)$$

Where

$$G_2 = \frac{\beta \cdot \eta \cdot \varepsilon \cdot b \cdot \psi}{AS}. \quad (3.2)$$

Hence, the first order conditions are as follows

$$\frac{\partial \mathcal{H}_1}{\partial W_1} = -\mathrm{e}^{-it}[\frac{W_1}{k} - \frac{g}{k} - C_0 - C_1H_1 - G_2(1-\alpha)] + \lambda_1[\frac{(\alpha-1)}{AS}] = 0. \quad (3.3)$$

$$\dot{\lambda}_1 = -\frac{\partial \mathcal{H}_1}{\partial H_1}. \quad (3.4)$$

$$\lambda_1^*(t_T, W_1^*(t_T), H_1^*(t_T)) = \lambda_2^*(t_T, W_2^*(t_T), H_2^*(t_T)). \quad (3.5)$$

$$H_1^*(t_T, W_1^*(t_T), H_1^*(t_T)) = \frac{\partial SP_2^*(t_T, W_1^*(t_T), H_1^*(t_T))}{\partial t_T}. \quad (3.6)$$

$$\dot{H}_1 = \frac{1}{AS}[R + (\alpha - 1)W_1]. \quad (3.7)$$

The transversality condition is given by $\lim_{t\to\infty}\lambda_1(t) = 0$. From Equation (3.3), we obtain the value for the costate variable $\lambda_1$ as follows.

$$\lambda_1 = \frac{1}{m}\mathrm{e}^{-it}[\frac{W_1}{k} - \frac{g}{k} - C_0 - C_1H_1 - G_2(1-\alpha)], \quad (3.8)$$

where $m = \frac{(\alpha-1)}{AS}$. The derivative of $\lambda_1$ with respect to $t$ is given by

$$\dot{\lambda}_1 = \frac{1}{m}\mathrm{e}^{-it}[-\frac{iW_1}{k} + \frac{ig}{k} + iC_0 + iC_1H_1 + iG_2(1-\alpha) - \frac{C_1R}{AS} - C_1mW_1 + \frac{\dot{W}_1}{k}]. \quad (3.9)$$

The derivative of $\mathcal{H}_1$ with respect to the water table elevation $H_1$ is given by

$$-\frac{\partial \mathcal{H}_1}{\partial H_1} = -C_1 W_1 \mathrm{e}^{-it}. \tag{3.10}$$

From Equation (3.4) and (3.9), we obtain the following equation.

$$-C_1 W_1 = \frac{1}{m}[-\frac{iW_1}{k} + \frac{ig}{k} + iC_0 + iC_1 H_1 + iG_2(1-\alpha) - \frac{C_1 R}{AS} - C_1 m W_1 + \frac{\dot{W}_1}{k}]. \tag{3.11}$$

Solving for $\dot{W}_1$ in the above equation we get the following equations.

$$\frac{\dot{W}_1}{mk} = \frac{iW_1}{mk} - \frac{ig}{mk} - \frac{iC_0}{m} - \frac{iC_1 H_1}{m} - \frac{iG_2(1-\alpha)}{m} + \frac{C_1 R}{mAS} + \frac{C_1 m W_1}{m} - C_1 W_1 \tag{3.12}$$

$$\frac{\dot{W}_1}{k} = \frac{iW_1}{k} - \frac{ig}{k} - iC_0 - iC_1 H_1 - iG_2(1-\alpha) + \frac{C_1 R}{AS} + C_1 m W_1 - C_1 m W_1 \tag{3.13}$$

$$\dot{W}_1 = iW_1 - ig - iC_0 k - iC_1 H_1 k - kiG_2(1-\alpha) + \frac{kC_1 R}{AS} \tag{3.14}$$

$$\dot{W}_1 = iW_1 - ikC_1 H_1 + [-ig - ikC_0 - ikG_2(1-\alpha) + \frac{C_1 kR}{AS}]. \tag{3.15}$$

Likewise, the value for $\dot{H}_1$ can be rewritten as

$$\dot{H}_1 = \frac{(\alpha-1)W_1}{AS} + \frac{R}{AS}. \tag{3.16}$$

Consequently, we now have to solve the two simultaneous differential equations ((3.15) and (3.16)). Thus, by letting $m = \frac{(\alpha-1)}{AS}$, $u = ikC_1$, $N = -ig - ikC_0 - ikG_2(1-\alpha) + \frac{C_1 kR}{AS}$ and $M = \frac{R}{AS}$, we get the following system of differential equations.

$$\dot{W}_1 = iW_1 - u \cdot H_1 + N. \tag{3.17}$$

$$\dot{H}_1 = m \cdot W_1 + M. \tag{3.18}$$

Putting the above system of differential equations in a $D$ operator format (where $D = \frac{d}{dt}$), and solving for $W_1$ yields the following second order linear non-homogeneous differential equation.

$$[(D^2 - Di) + u \cdot m]W_1 = -u \cdot M. \tag{3.19}$$

The particular solution of the above differential equation is given by: $-\frac{M}{m}$ and the characteristic roots by $y_{1,2} = \frac{i \pm \sqrt{i^2 - 4um}}{2}$. Furthermore, the steady state level water table is given by

$$H_1^* = [\frac{-i\frac{M}{m} + N}{u}] \tag{3.20}$$

Hence, the solution for $W_1^*(t)$ and $H_1^*(t)$ is given by

$$W_1^*(t) = \tilde{A}e^{y_1 t} + \tilde{B}e^{y_2 t} - \frac{M}{m}. \tag{3.21}$$

$$H_1^*(t) = \frac{m}{y_1}\tilde{A}e^{y_1 t} + \frac{m}{y_2}\tilde{B}e^{y_2 t} + \frac{N - i\frac{M}{m}}{u}. \tag{3.22}$$

Where $\tilde{A}$ and $\tilde{B}$ are obtained by imposing the initial conditions.

$$\tilde{B} = \frac{y_2 AS}{\alpha - 1}[H_0 - \frac{N - \frac{iR}{\alpha - 1}}{ikC_1} - \frac{[H_T - \frac{N - \frac{iR}{\alpha - 1}}{ikC_1}] - [H_0 - \frac{N - \frac{iR}{\alpha - 1}}{ikC_1}]e^{y_2 t_T}}{e^{y_1 t_T} - e^{y_2 t_T}}], \tag{3.23}$$

$$\tilde{A} = \frac{y_1 AS}{\alpha - 1}[\frac{[H_T - \frac{N - \frac{iR}{\alpha - 1}}{ikC_1}] - [H_0 - \frac{N - \frac{iR}{\alpha - 1}}{ikC_1}]e^{y_2 t_T}}{e^{y_1 t_T} - e^{y_2 t_T}}]. \tag{3.24}$$

The maximization principle specifies the necessary conditions for optimality. However, it is also necessary to ensure that the second-order conditions are met. The compliance of the second order conditions ensures that the maximum principle's necessary conditions are likewise sufficient for global optimality. Mangasarian established a basic sufficiency theorem (Chiang 1992, pp. 214–217) that guarantees the second order conditions. In this problem, the sufficient conditions of the Mangasarian theorem have been verified, allowing us to conclude that the obtained trajectories are optimal.

## Appendix 4. Proof of Proposition (1)

To determine the impact of land sinking on the optimal solutions, we differentiate the expressions for the water table and extractions with respect to the economic cost of land sinking.

$$\frac{\partial W(t)}{\partial \beta} = -\eta \cdot \varepsilon \cdot b \cdot \psi \cdot \frac{x_2 \Omega (1-\alpha)}{(\alpha-1) C_1 G_6} \mathrm{e}^{x_2 (t-t_T)}. \quad (4.1)$$

We know that $\eta > 0$, $\Omega > 0$, $b > 0$, $\mathrm{e}^{x_2(t-t_T)} > 0$, $\psi > 0$, $k < 0$, $C_1 < 0$, $G_6 < 0$, $(1-\alpha) > 0$, $(\alpha - 1) < 0$, and $\varepsilon > 0$ since an increase in the confining unit material or a compacting sediment induces a reduction in it's volume. If there was no $x_2$, the derivative's sign would be positive. Therefore, the sign of the derivative depends on the value of $x_2$. If $i < \sqrt{i^2 - \frac{ikC_1(\alpha-1)}{AS}}$, the sign of the derivative is negative. If $i > \sqrt{i^2 - \frac{ikC_1(\alpha-1)}{AS}}$, the sign of the derivative is positive, but this case can not occur. This is because $-\frac{ikC_1(\alpha-1)}{\Omega AS} > 0$ and hence $i$ is always less than $\sqrt{i^2 - \frac{ikC_1(\alpha-1)}{\Omega AS}}$.

$$\frac{\partial H(t)}{\partial \beta} = \frac{\eta \cdot \varepsilon \cdot b \cdot \psi \cdot (1-\alpha)}{C_1 G_6 AS} \cdot [1 - \mathrm{e}^{x_2(t-t_T)}]. \quad (4.2)$$

In this case, if there was no $(1 - \mathrm{e}^{x_2(t-t_T)})$, the derivative's sign would be positive. Therefore, the sign of the derivative depends on the value of $(1 - \mathrm{e}^{x_2(t-t_T)})$. If $\mathrm{e}^{x_2(t-t_T)} > 1$, the sign of the derivative is negative but this case can not occur because $x_2$ is negative. If $\mathrm{e}^{x_2(t-t_T)} < 1$, the sign of the derivative is positive.

**Appendix 5. Proof of Proposition (2)**

To determine the impact of the aquifer storage capacity reduction on the optimal solutions, we differentiate the expression for the economic cost ($\phi(W,H)$) of losing the aquifer's storage capacity with respect to the optimal water table height and extractions, respectively.

$$\frac{\partial \phi(W^*, H^*)}{\partial W^*} = \frac{1}{k}. \quad (5.1)$$

Since $k < 0$, the derivative's sign is negative. Therefore, the higher the optimal level of extractions the lower the Pigouvian tax. In other words, the higher the Pigouvian tax the lower the optimal level of extractipns.

$$\frac{\partial \phi(W^*, H^*)}{\partial H^*} = -C_1. \quad (5.2)$$

Since $C_1 < 0$, the derivative's sigh is positive. Therefore, the higher the Pigouvian tax the higher the optimal level of the water table.

**Appendix 6. Proof of Proposition (3)**

To determine the impact of land sinking on the optimal solutions, we differentiate the expressions for the water table and extractions with respect to the economic cost of land sinking.

$$\frac{\partial W(t)}{\partial \beta} = \frac{\eta \cdot \varepsilon \cdot b \cdot \psi \cdot (1-\alpha)}{C_1 AS} \cdot \frac{(\alpha-1) y_2}{AS(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})} \mathrm{e}^{y_2 t_T + y_2 t}. \tag{6.1}$$

We know that $\eta > 0$, $b > 0$, $A > 0$, $S > 0$, $\psi > 0$, $C_1 < 0$, $(\alpha - 1) < 0$, $(1 - \alpha) > 0$ and $\varepsilon > 0$ since an increase in the confining unit material or a compacting sediment induces a reduction in it's volume. If there was no $y_2$ and $(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})$, the derivative's sign would be positive. Therefore, the sign of the derivative depends on the value of $y_2$ and $(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})$. If $i < \sqrt{i^2 - \frac{ikC_1(\alpha-1)}{AS}}$ and $(\mathrm{e}^{y_1 t_T} > \mathrm{e}^{y_2 t_T})$, the sign of the derivative is negative. However, this is the only case that can occur since $y_2 < 0$ and $y_1 > 0$.

$$\frac{\partial H(t)}{\partial \beta} = \frac{\eta \cdot \varepsilon \cdot b \cdot \psi \cdot (1-\alpha)}{C_1 AS} \cdot \left[\frac{\mathrm{e}^{y_2 t_T + y_2 t} (\alpha-1)^2}{(AS)^2 (\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})} - 1\right]. \tag{6.2}$$

In this case, if there was no $\left[\frac{\mathrm{e}^{y_2 t_T + y_2 t}(\alpha-1)^2}{(AS)^2(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})} - 1\right]$, the derivative's sign would be negative. Therefore, the sign of the derivative depends on the value of $(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})$ and $\frac{\mathrm{e}^{y_2 t_T + y_2 t}(\alpha-1)^2}{(AS)^2(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})}$. If $\mathrm{e}^{y_1 t_T} > \mathrm{e}^{y_2 t_T}$, the sign of the derivative depends on the value of $\frac{\mathrm{e}^{y_2 t_T + y_2 t}(\alpha-1)^2}{(AS)^2(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})}$. However, this is the only case that can occur since $y_2 < 0$ and $y_1 > 0$. Therefore, if $\frac{\mathrm{e}^{y_2 t_T + y_2 t}(\alpha-1)^2}{(\mathrm{AS})^2(\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T})} < 1$, the sign of the derivative is positive.

## Appendix 7. Detailed solution of the quotas optimal control problem

When both the economic costs attached to mitigating LS impacts are equal to zero and the storage externality constant representing the LS impact on aquifer storage capacity is equal to 1, the optimal path for groundwater extractions is given as follows (see Gisser and Sanchez, 1980)

$$W^{\star}(t) = \frac{y_2 AS}{\alpha - 1}\left[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}\right]\mathrm{e}^{y_2 t} - \frac{R}{\alpha - 1}, \tag{7.1}$$

Where $N_0 = \frac{kC_1R}{AS} - ig - ikC_0$. Using equation (7.1), we determine the value of $N_0$ that satisfies the condition $W^{\star}(t) \leq \widehat{W}$.

$$\frac{y_2AS}{\alpha-1}[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2t} - \frac{R}{\alpha-1} \leq \widehat{W} \tag{7.2}$$

$$\frac{y_2AS}{\alpha-1}[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2t} \leq \frac{\widehat{W}(\alpha-1)+R}{\alpha-1} \tag{7.3}$$

$$[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2t} \leq \frac{\widehat{W}(\alpha-1)+R}{y_2AS} \tag{7.4}$$

$$[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}] \leq \frac{\widehat{W}(\alpha-1)+R}{y_2AS}\mathrm{e}^{-y_2t} \tag{7.5}$$

$$H_0 \cdot ikC_1 - \frac{\widehat{W}(\alpha-1)+R}{y_2AS}\mathrm{e}^{-y_2t} \cdot ikC_1 \leq N_0 - \frac{iR}{\alpha-1} \tag{7.6}$$

$$H_0 \cdot ikC_1 - \frac{\widehat{W}(\alpha-1)+R}{y_2AS}\mathrm{e}^{-y_2t} \cdot ikC_1 + \frac{iR}{\alpha-1} \leq N_0 \tag{7.7}$$

If we let the LHS of Equation(7.7) to be equal to $N_A(t)$, we then obtain

$$W^{\star}(t)=\begin{cases}\frac{y_2 AS}{\alpha-1}[H_0-\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1}]e^{y_2 t}-\frac{R}{\alpha-1} & N_0\geq N_A(t)\\ \widehat{W} & N_0<N_A(t)\end{cases} \tag{7.8}$$

When $W^{\star}(t)=\widehat{W}$, we equate the RHS of Equation(7.1) to $\widehat{W}$. We obtain that (solving for $N_0$) the latter is only satisfied if $N_0$ is equal to $N_A(t)$. Hence, the corresponding water table path should also satisfy this condition.

$$H^{\star}(t)=\begin{cases}[H_0-\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1}]e^{y_2 t}+\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1} & N_0\geq N_A(t)\\ [H_0-\frac{N_A(t)-i\frac{R}{\alpha-1}}{ikC_1}]e^{y_2 t}+\frac{N_A(t)-i\frac{R}{\alpha-1}}{ikC_1} & N_0<N_A(t)\end{cases} \tag{7.9}$$

The conditions to ensure that a maximum has been achieved have been verified.

**Appendix 8. Proof of Proposition (4)**

Recall that the optimal paths for groundwater extractions and water table level under the quota control problem are given as follows.

$$W^{\star}(t)=\begin{cases}\frac{y_2 AS}{\alpha-1}[H_0-\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1}]e^{y_2 t}-\frac{R}{\alpha-1} & N_0\geq N_A(t)\\ \widehat{W} & N_0<N_A(t)\end{cases} \tag{8.1}$$

$$H^{\star}(t)=\begin{cases}[H_0-\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1}]e^{y_2 t}+\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1} & N_0\geq N_A(t)\\ [H_0-\frac{N_A(t)-i\frac{R}{\alpha-1}}{ikC_1}]e^{y_2 t}+\frac{N_A(t)-i\frac{R}{\alpha-1}}{ikC_1} & N_0<N_A(t)\end{cases} \tag{8.2}$$

$$N_0=\frac{kC_1R}{AS}-ig-ikC_0, \tag{8.3}$$

$$N_A(t) = H_0 \cdot ikC_1 - \frac{[\widehat{W}(\alpha-1)+R]ikC_1 \mathrm{e}^{-y_2 t}}{y_2 AS} + \frac{iR}{\alpha-1}, \tag{8.4}$$

$$y_2 = \frac{i-\sqrt{i^2-4\cdot\frac{ikC_1(\alpha-1)}{AS}}}{2}, \quad y_2 < 0. \tag{8.5}$$

We observe that the case when $N_0 < N_A(t)$ occurs first during the planning period since $\frac{kC_1R}{AS} - ig - ikC_0 < H_0 \cdot ikC_1 - \frac{[\widehat{W}(\alpha-1)+R]ikC_1 \mathrm{e}^{-y_2 t}}{y_2 AS} + \frac{iR}{\alpha-1}$ for values of $t$ starting from time $t = 0$ up to a certain time $t$ during the planning period at which $\mathrm{e}^{-y_2 t}$ converges to positive $\infty$ and $N_A(t)$ becomes greater than or equal to $N_0$. Therefore, the case $N_0 \geq N_A(t)$ occurs second (lastly) during the planning period. Of course we observe that the sign of the term $\frac{[\widehat{W}(\alpha-1)+R]ikC_1 \mathrm{e}^{-y_2 t}}{y_2 AS}$ would be negative if there was no $[\widehat{W}(\alpha-1)+R]$ present since $ikC_1 > 0$, $\mathrm{e}^{-y_2 t} > 0$, and $y_2 AS < 0$. We need $\left[\widehat{W}(\alpha-1)+R\right] < 0$ such that $N_A(t)$ becomes lower than or equal to $N_0$ when $\mathrm{e}^{-y_2 t}$ converges to positive $\infty$. Intuitively, $\left[\widehat{W}(\alpha-1)+R\right] < 0$ implies that $R < \widehat{\mathrm{W}} - \widehat{\mathrm{W}}\alpha$. That is, the term $[\widehat{W}(\alpha-1)+R]$ will only be negative if the aquifer's recharge is less than the specified extraction level (quota level) minus return flows to the aquifer, which should always be the case for quotas to be applicable. Otherwise there is no need to apply quotas if $R > \widehat{\mathrm{W}} - \widehat{\mathrm{W}}\alpha$ since there is no over-extraction happening. This rules out the case that the term $\left[\widehat{W}(\alpha-1)+R\right]$ can also have a positive sign.

## Appendix 9. Proof of Proposition (5)

To determine the impact of the quota level on the optimal solutions, we differentiate the expressions for the extractions with respect to the quota level.

$$\frac{\partial W^{*}(t)}{\partial \widehat{W}} = \begin{cases} 0 & N_0 \geq N_A(t) \\ 1 & N_0 < N_A(t) \end{cases} \tag{9.1}$$

Intuitively, When $N_0 < N_A(t)$ (first phase of the planning period), the higher the quota level the higher the optimal level of extractions. When $N_0 \geq N_A(t)$, increasing the quota level has no effect on groundwater extractions.

## Appendix 10. Proof of Proposition (6)

To determine the impact of the quota level on the optimal solutions, we differentiate the expressions for the water table level with respect to the quota level.

$$\frac{\partial H^{\star}(t)}{\partial \widehat{W}} = \begin{cases} 0 & N_0 \geq N_A(t) \\ \frac{(\alpha-1)}{y_2 AS}\left[\mathrm{e}^{-y_2 t} - 1\right] & N_0 < N_A(t) \end{cases} \tag{10.1}$$

We know that $(\alpha - 1) < 0$, $A > 0$, $S > 0$, $k < 0$, and $C_1 < 0$. If there was no $y_2$ and $[\mathrm{e}^{-y_2 t} - 1]$, the derivative's sign would be negative. Therefore, the sign of the derivative depends on the value of $y_2$ and $[\mathrm{e}^{-y_2 t} - 1]$. Intuitively, the range of $\mathrm{e}^{-y_2 t}$ is equal to $(0, \infty)$ since $y_2 < 0$, and the range of $[\mathrm{e}^{-y_2 t} - 1]$ is equal to $(-1, \infty)$. Therefore, if $i < \sqrt{i^2 - 4\frac{ikC_1(\alpha-1)}{AS}}$ and $\mathrm{e}^{-y_2 t} < 1$, the sign of the derivative is negative. If $i < \sqrt{i^2 - 4\frac{ikC_1(\alpha-1)}{AS}}$ and $\mathrm{e}^{-y_2 t} > 1$, the sign of the derivative is positive. Otherwise, if $i < \sqrt{i^2 - 4\frac{ikC_1(\alpha-1)}{AS}}$ and $\mathrm{e}^{-y_2 t} = 1$ (which is only possible at time $t = 0$), the derivative sign is equal to zero. However, these are the only cases that can occur since $y_2 < 0$.

## Appendix 11. Detailed solution of the packaging and sequencing optimal control problem

Intuitively, the optimal solution to the maximization problem (6) and (25) -(29) should have two solutions, the first solution applies when $W(t) \leq \widehat{W}$ (and quota restriction applies), the second solution is when $W(t) > \widehat{W}$ (when the tax policy applies). Both of the optimal solutions were obtained already in the previous proofs. For the quotas option, we obtained the following optimal solution

$$W^{\star}(t) = \begin{cases} \frac{y_2 AS}{\alpha-1}\left[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}\right]\mathrm{e}^{y_2 t} - \frac{R}{\alpha-1} & N_0 \geq N_A(t) \\ \widehat{W} & N_0 < N_A(t) \end{cases} \tag{11.1}$$

$$H^{\star}(t) = \begin{cases} [H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2 t} + \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1} & N_0 \geq N_A(t) \\ [H_0 - \frac{N_A(t) - i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2 t} + \frac{N_A(t) - i\frac{R}{\alpha-1}}{ikC_1} & N_0 < N_A(t) \end{cases} \tag{11.2}$$

$$N_0 = \frac{kC_1 R}{AS} - ig - ikC_0, \tag{11.3}$$

$$N_A(t) = H_0 \cdot ikC_1 - \frac{\widehat{W}(\alpha-1)+R}{y_2 AS} e^{-y_2 t} \cdot ikC_1 + \frac{iR}{\alpha-1}, \tag{11.4}$$

$$y_2 = \frac{i - \sqrt{i^2 - 4 \cdot \frac{ikC_1(\alpha-1)}{AS}}}{2}, \quad y_2 < 0. \tag{11.5}$$

While for the tax policy option, we obtained the following optimal solution

$$W^{\star}(t) = \begin{cases} \tilde{A}\mathrm{e}^{y_1 t} + \tilde{B}\mathrm{e}^{y_2 t} - \frac{R}{\alpha-1} & t < t_T \\ \frac{\Omega \cdot x_2 AS}{\alpha-1}[H_T - (\frac{iR}{\alpha-1} - NN)\frac{G_7}{ikC_1 G_6}]\mathrm{e}^{x_2(t-t_T)} - \frac{R}{\alpha-1} & t \geq t_T \end{cases} \tag{11.6}$$

$$H^{\star}(t) = \begin{cases} \frac{(\alpha-1)}{AS\, y_1}\tilde{A}\mathrm{e}^{y_1 t} + \frac{(\alpha-1)}{AS\, y_2}\tilde{B}\mathrm{e}^{y_2 t} + \frac{N - \frac{iR}{\alpha-1}}{ikC_1} & t < t_T \\ [H_T - (\frac{iR}{\alpha-1} - NN)\frac{G_7}{ikC_1 G_6}]\mathrm{e}^{x_2(t-t_T)} + (\frac{iR}{\alpha-1} - NN)\frac{G_7}{ikC_1 G_6} & t \geq t_T, \end{cases} \tag{11.7}$$

where

$$x_2 = \frac{i - \sqrt{i^2 + 4 \cdot \frac{ikC_1 G_6(\alpha-1)}{G_7 \Omega \cdot AS}}}{2}, \quad x_2 < 0, \tag{11.8}$$

$$NN = \frac{1}{G_7}[-ig - ikC_0 + ikG_5 - \frac{kG_6 C_1 R}{\Omega \cdot AS} - \frac{mk}{\Omega} G_3 R C_1], \quad (11.9)$$

$$G_6 = G_3(1-\alpha) - 1. \quad (11.10)$$

$$G_7 = 1 - 2G_3(1-\alpha). \quad (11.11)$$

$$G_2 = \frac{\beta \cdot \eta \cdot \varepsilon \cdot b \cdot \psi}{AS}. \quad (11.12)$$

$$G_3 = b\psi\gamma_w(1 - n + n_w). \quad (11.13)$$

$$G_5 = \frac{RG_3}{k} + \frac{(1-\alpha)gG_3}{k} + G_3(1-\alpha)C_0 - G_2(1-\alpha). \quad (11.14)$$

$$y_{1,2} = \frac{i \pm \sqrt{i^2 - 4 \cdot \frac{ikC_1(\alpha-1)}{AS}}}{2}, \quad (11.15)$$

$$N = -ig - ikC_0 - ikG_2(1-\alpha) + \frac{kC_1 R}{AS}, \quad (11.16)$$

$$\tilde{B} = \frac{y_2 AS}{\alpha - 1}[H_0 - \frac{N - \frac{iR}{\alpha-1}}{ikC_1} - \frac{[H_T - \frac{N - \frac{iR}{\alpha-1}}{ikC_1}] - [H_0 - \frac{N - \frac{iR}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2 t_T}}{\mathrm{e}^{y_1 t_T} - \mathrm{e}^{y_2 t_T}}], \quad (11.17)$$

$$\tilde{A} = \frac{y_1 AS}{\alpha-1}\left[\frac{[H_T - \frac{N-\frac{iR}{\alpha-1}}{ikC_1}] - [H_0 - \frac{N-\frac{iR}{\alpha-1}}{ikC_1}]e^{y_2 t_T}}{e^{y_1 t_T} - e^{y_2 t_T}}\right], \tag{11.18}$$

When no policy on quotas or taxes is in place, the optimal path for groundwater extractions is given as follows (see Gisser and Sanchez, 1980)

$$W^{\star}(t) = \frac{y_2 AS}{\alpha-1}\left[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}\right]e^{y_2 t} - \frac{R}{\alpha-1}, \tag{11.19}$$

Where $N_0 = \frac{kC_1 R}{AS} - ig - ikC_0$. Taking the limit of $W^{\star}(t)$ in equation (11.19) as $t$ goes to infinity yields $-\frac{R}{\alpha-1} > 0$, where $-\frac{R}{\alpha-1} > 0$ is the steady state solution for $W^{\star}(t)$. Intuitively, since $W^{\star}(t) > 0$ then $W^{\star}(t=0) > -\frac{R}{\alpha-1}$. That is, $\frac{y_2 AS}{\alpha-1}[H_0 - \frac{N_0 - i\frac{R}{\alpha-1}}{ikC_1}]e^{y_2 t} - \frac{R}{\alpha-1} > -\frac{R}{\alpha-1}$ since $i > 0$, $g > 0$, $k < 0$, $C_0 > 0$, $C_1 < 0$, and $H_0 > 0$. Theoretically, the optimal extraction levels should start at a level higher than steady state level (baseline scenario) in year zero and continue rising as population and economic activities increases over time. At the end, as $t$ goes to infinity, the extraction levels should decrease as the height of the water table reduces which makes extraction costs costly and the steady state will be reached. As a result, the extraction levels that are higher than the quota level $\widehat{W}$ should fall in the first phase of the planning period ($t < t_T$), while those lower than the quota level should fall in the second phase of the planning period ($t \geq t_T$). Thus, the policy on taxes is applied first, and as the extraction levels start to be less than or equal to the quota level, then the quota policy is applied. This is because the recharge rate is assumed constant in our model. Intuitively, when $W^{\star}(t) = \widehat{W}$ in equation (11.1), then extractions are higher than the quota level, the tax policy should be applied. Therefore, in the optimal solution for quotas, we substitute $\widehat{W}$ for the optimal extraction levels when the tax policy is applied. Combining the optimal solutions for quotas and taxes gives the optimal solution for the combination of the two policies as follows below.

$$H^{\star}(t)=\begin{cases}\frac{(\alpha-1)}{ASy_1}\tilde{A}\mathrm{e}^{y_1 t}+\frac{(\alpha-1)}{ASy_2}\tilde{B}\mathrm{e}^{y_2 t}+\frac{N-\frac{iR}{\alpha-1}}{ikC_1} & t<t_T\\ [H_0-\frac{N_A(t)-i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2(t-t_T)}+\frac{N_A(t)-i\frac{R}{\alpha-1}}{ikC_1} & t\geq t_T\ \&\ N_0<N_A(t)\\ [H_0-\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2(t-t_T)}+\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1} & t\geq t_T\ \&\ N_0\geq N_A(t),\end{cases} \tag{11.20}$$

$$W^{\star}(t)=\begin{cases}\tilde{A}\mathrm{e}^{y_1 t}+\tilde{B}\mathrm{e}^{y_2 t}-\frac{R}{\alpha-1} & \mathrm{t}<t_T\\ \widehat{W} & t\geq t_T\ \&\ N_0<N_A(t)\\ \frac{y_2 AS}{\alpha-1}[H_0-\frac{N_0-i\frac{R}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2 t}-\frac{R}{\alpha-1} & t\geq t_T\ \&\ N_0\geq N_A(t)\end{cases} \tag{11.21}$$

where

$$N_0=\frac{kC_1R}{AS}-ig-ikC_0, \tag{11.22}$$

$$N_A(t)=H_0\cdot ikC_1-\frac{\widehat{W}(\alpha-1)+R}{y_2AS}\mathrm{e}^{-y_2 t}\cdot ikC_1+\frac{iR}{\alpha-1}, \tag{11.23}$$

$$y_2=\frac{i-\sqrt{i^2-4\cdot\frac{ikC_1(\alpha-1)}{AS}}}{2},\quad y_2<0, \tag{11.24}$$

$$y_{1,2}=\frac{i\pm\sqrt{i^2-4\cdot\frac{ikC_1(\alpha-1)}{AS}}}{2}, \tag{11.25}$$

$$N=-ig-ikC_0-ikG_2(1-\alpha)+\frac{kC_1R}{AS}, \tag{11.26}$$

$$\tilde{B} = \frac{y_2 AS}{\alpha-1}\left[H_0 - \frac{N-\frac{iR}{\alpha-1}}{ikC_1} - \frac{[H_T - \frac{N-\frac{iR}{\alpha-1}}{ikC_1}] - [H_0 - \frac{N-\frac{iR}{\alpha-1}}{ikC_1}]e^{y_2 t_T}}{e^{y_1 t_T} - e^{y_2 t_T}}\right], \quad (11.27)$$

$$\tilde{A} = \frac{y_1 AS}{\alpha-1}\left[\frac{[H_T - \frac{N-\frac{iR}{\alpha-1}}{ikC_1}] - [H_0 - \frac{N-\frac{iR}{\alpha-1}}{ikC_1}]e^{y_2 t_T}}{e^{y_1 t_T} - e^{y_2 t_T}}\right]. \quad (11.28)$$

## Appendix 12. Application to the Dendron aquifer system (Additional information on data for the numerical application)

The area had around 335 boreholes in 1986, with irrigation accounting for 95% of groundwater withdrawals (Jolly, 1986). The remaining 5% of groundwater withdrawals were for domestic consumption and livestock watering. According to Masiyandima et al. (2002), between 1968 and 1986, the farmers' union set a regulation that only 3% of each 1000 hectares of land should be irrigated with groundwater, in an attempt to prevent overexploitation of the aquifer system. In addition, farmers began practicing a variety of cropping patterns and irrigation water management strategies, such as switching from furrow irrigation to manual move sprinkler systems, and finally center pivots, which are now utilized on the majority of farms in the area (Masiyandima et al., 2002). As a result, around early 1990s, water table levels began to rise again in the aquifer system. Severe flood events, in combination with the aforementioned farming patterns and irrigation water management practices, induced a rise in the water table level. Severe flood events have been observed in the Limpopo River Basin in the last ten years, in 1955, 1967, 1972, 1975, 1977, 1981, 1990, 2000, and 2013 (CRIDF, 2018). The height of the water table decreased at a much higher rate in the year 2000, resulting in a water table height range of roughly 1239.5 to 1189.5 meters above sea level (Masiyandima et al., 2002). Table 1. shows groundwater drawdown and the height of the water table levels in the Dendron aquifer system over the years for which data is available.

**Table 1**. Groundwater drawdown and height of the water table levels in the Dendron aquifer system.

| Year | Groundwater level drawdown (m) | Water table height (m.a.s.l) |
|---|---|---|
| 1968 | 9 | 1277.5 – 1268.5 |
| 1969 | 1.5 | 1271.5 |
| 1974 | 17 | 1274.5 -1254.5 |
| 1976 | 5 | 1259.5 |
| 1986 | 13 | 1246.5 |
| 2000 | 57 | 1239.5 – 1189.5 |

Source: Jolly (1986); Masiyandima et al., (2002)

The hydrological and economic data used in our empirical application are collected from previous studies in the area as well as groundwater reports from the South Africa Department of Water and Sanitation (DWS). The Hout River Catchment is characterized by a semi-arid climate, with an average annual rainfall of 407 millimeters, sandy soil, with Luvisols covering approximately 56% of the catchment (Ebrahim et al., 2019). Geologically, the Dendron aquifer system is made up of two interdependent aquifer system, the upper weathered granite aquifer system and the lower fractured granite aquifer system (Jolly, 1986). In other words, the aquifer system is made up of a first aquifer system comprised of weathered bedrocks (weathered zone) that sits on top of a lower aquifer system made of fractured rocks (fractured zone). Archaen rocks, which include leucocratic granites and gneisses, make up the aquifer system (Jolly, 1986). The weathered zone is said to be unconfined, whereas the fractured zone is. According to Murray and Tredoux (2002), the two aquifer systems are partly infilled with clay sediments as a result of weathering in the upper aquifer system. The presence of fine-grained sediments (clay sediments) within the aquifer system makes the Dendron aquifer more vulnerable to LS episodes.

The storage capacity of the aquifer system is estimated at 124 million cubic meters. The landscape is mostly flat. The upper aquifer system water table is reported to be between 1277.5-1239.5 meters above sea level, while the lower aquifer system extends up to 1169.5 meters above sea level. There is little groundwater at heights below 1169.5 meters above sea level (Jolly, 1986). The connection between the upper and lower aquifer system, according to Masiyandima et al. (2002), is at 1254.5-1239.5 meters above sea level. In comparison to the lower aquifer system, which has a high yield, the upper aquifer system has a low storage and yield (Jolly, 1986). According to Holland (2012), the upper aquifer system has dried up in most areas of the aquifer, leaving just the lower aquifer with water. As a result, the majority of agricultural production wells in the aquifer system are drilled in the aquifer system's lower zone, which is the lower fractured aquifer system (Fallon et al., 2018; Holland 2011, 2012). Therefore, the yield-related parameters for the aquifer in our analysis are determined using the lower aquifer system's hydrological data. We only simulate the lower aquifer because all of the parameters we used are for the lower aquifer system, where boreholes are currently drilled, while the upper aquifer system has run dry owing to over-exploitation. However, we take the whole aquifer thickness of the entire aquifer system into account, which includes both the lower and higher aquifer systems. When certain yield-related parameters are unavailable, other authors have used values from other weathered-fractured aquifer systems to analyze groundwater in the Dendron area in the past (Jolly, 1986; Ebrahim et al., 2019). We use the same approach.

The economic prices and cost values are expressed in 2011 US dollars. The price of irrigation water is 3907.38 US dollars (27, 000 Rands) per million cubic meters, based on the 2011 currency rate (Lange and Hassan, 2006). This figure represents the average tariff for raw water in the catchment area. We calculate the intercept of the demand function to be 62 using the average groundwater abstractions from the aquifer system of 17 million cubic meters per year (Ebrahim et al., 2019). According to DWAF (2003), the fixed pumping cost in a fractured rock aquifer system with a water table height below 1169.5 meters above sea level in South Africa is 4, 551.20 US dollars per year. This figure represents the fixed cost of operating and maintaining a pump (or borehole), that is, the cost when no groundwater is pumped. This covers mechanical and electrical maintenance, as well as the amortization of extraction technology. The electrical expenses to pump water from the aquifer system are estimated to be 0.0026 USD per cubic meter or 2,604.92 US dollars per million cubic meters (DWAF, 2003). To maintain the same reference year for the parameter values as 2011, the pumping cost intercept in 2011 is 5,209.84 US dollars. Using the pumping cost function, we find that the slope of the pumping cost function in the area is -3.94.

**Table 2**: Hydrological and economic values of the Dendron aquifer system.

| Parameter | Description | Units | Value | Source |
|---|---|---|---|---|
| $k$ | Water demand slope | $\$/Mm^3$ | -0.0425 | Authors |
| $g$ | Water demand intercept | $\$/Mm^3$ | 62 | Authors |
| $C_0$ | Pumping costs intercept | $\$/Mm^3$ | 5209.84 | Authors |
| $C_1$ | Pumping costs slope | $\$/Mm^3\ m$ | -3.94 | Authors |
| $\alpha$ | Return flow coefficient | dimensionless | 0.2 | Jolly (1986) |
| $H_0$ | Current water table | $m$ | 1191.5 | Fallon et al. (2018) |
| $H_T$ | Critical water table level | $m$ | 1189.5 | Authors |
| $R$ | Natural recharge | $Mm^3/year$ | 7.35 | Jolly (1986) |
| $A$ | Aquifer system area | $km^2$ | 1600 | Masiyandima et al. (2002) |
| $S$ | Storativity coefficient | dimensionless | 0.0025 | Masiyandima et al. (2002) |
| $i$ | Social discount rate | % | 0.08 | Conningarth Economists (2014, pp.69-70). |
| $\beta$ | Pigouvian tax per unit of land sinking | $\$/\mathrm{m}$ | 1,245 | Authors |
| $\eta$ | Water density | $Kg/m^3$ | 1000 | Wade et al. (2018) |
| $b$ | Aquifer system's thickness | $m$ | 110 | Masiyandima et al. (2002) |
| $\psi$ | Aquifer system's compressibility | $ms^2/kg$ | $5.1 \times 10^{-10}$ | Authors |
| $n$ | Porosity | dimensionless | 0.34 | Woessner and Poeter (2020) |
| $\varepsilon$ | Gravitational acceleration | $m/s^2$ | 9.81 | Wade et al. (2018) |
| $n_w$ | Vadose moisture/ Total volume | dimensionless | 0.1 | Jolly (1986) |

| | | | | |
|---|---|---|---|---|
| $\gamma_w$ | Unit weight of water | $N/m^3$ | 9810 | Poland and Davis (1969) |

## Appendix 13. Regulatory policies and their combined effects

### 13.1 LS and taxes

Dinar et al. (2021) developed an indicative index (LS Impact Magnitude (LSIE)) to quantify the extent of LS impact in locations around the world. The more intense the LS impact in that site, the higher the LSIE value. The LSIE index value for a site in the Western Cape province of South Africa is 0.6, making it the only South African site included in Dinar et al. (2021) LSIE index analysis. Without losing generality, this shows that the Limpopo province of South Africa is vulnerable to LS impacts. The storage externality constant representing the LS impact on aquifer system storage capacity is assumed to be $\Omega = 1 - \text{LSIE} = 1 - 0.6 = 0.4$. This is because the smaller the LS impact on the aquifer system storage capacity, the larger the constant $\Omega$ is. For the sensitivity analysis, we analyze the case when the observed LSIE index value reduces to 0.51 which results in the storage externality constant being $\Omega = 1 - \text{LSIE} = 1 - 0.51 = 0.49$. This gives us directions of how the optimal trajectories are when the aquifer system is less affected by land sinking. Once calibrated through simulation, we found that the water table level reduces and then rise (during the first phase), and sharply reduces (during the second phase). This is true for any storage externality constant ($\Omega$) in the range $0.4 < \Omega \leq 0.49$. And extractions also follow the same pattern. This indicates that the appropriate storage externality constant should be in that range.

Once calibrated through simulation, we found that, both the extractions and water table level only change significantly when the tax rate per unit of land sinking is strictly very higher. This is because the Dendron aquifer system is not highly compressible (aquifer system's compressibility equal to $\psi = 0.00000000051 \;\; ms^2/kg$) which indicates the aquifer system is less prone to LS impacts. As a result, we found, through calibrated simulations, that the tax rate per unit of land sinking should be above 4 million US Dollars in order to effect significant changes in both extractions and the water table level.

The aquifer system's archaen rocks distort (fracturing, faulting, and folding) as a result of weathering and unloading caused by erosion of overlying layers (Kelbe and Rawlins, 2004). This deformation generally occurs up to 1239.5 m.a.s.l (50 meters below irrigation surface), where the lower aquifer system began (Kelbe and Rawlins, 2004). Given that the lower aquifer is also deforming (fracturing), we assume, without loss of generality, that inelastic compaction begins if deformation extends for an additional 50 meters (at 1189.5 m.a.s.l).

**Appendix 14. Proof of the case when there is LS but no policy ineterventions.**

The hamiltonian function of the system (6)-(8) and a constraint ($0 < \Omega \leq 1,$ and $\beta = \phi(W, H) = 0$) in the second phase is given as follows

$$\mathcal{H}_2(t, W_2, H_2, \lambda_2) = -\mathrm{e}^{-it}[\frac{W_2^2}{2k} - \frac{gW_2}{k} - (C_0 + C_1 H_2)W_2] + \lambda_2 \cdot \frac{[R+(\alpha-1)W_2]}{\Omega \cdot AS} \tag{14.1}$$

Hence, the first order conditions are as follows

$$\frac{\partial \mathcal{H}_2}{\partial W_2} = -\mathrm{e}^{-it}[\frac{W_2}{k} - \frac{g}{k} - C_0 - C_1 H_2] + \lambda_2[\frac{(\alpha-1)}{\Omega \cdot AS}] = 0. \tag{14.2}$$

$$\dot{\lambda}_2 = -\frac{\partial \mathcal{H}_2}{\partial H_2}. \tag{14.3}$$

$$\dot{H}_2 = \frac{1}{\Omega \cdot AS}[R + (\alpha - 1)W_2]. \tag{14.4}$$

The transversality condition is given by $\lim_{t\to\infty}\lambda_2(t) = 0$. From Equation (14.2), we obtain the value for the costate variable $\lambda_2$ as follows.

$$\lambda_2 = \frac{\Omega}{m}\mathrm{e}^{-it}[(\frac{1}{k})W_2 - \frac{g}{k} - C_0 - C_1 H_2], \tag{14.5}$$

where $m = \frac{(\alpha-1)}{AS}$. The derivative of $\lambda_2$ with respect to $t$ is given by

$$\dot{\lambda}_2 = \frac{\Omega}{m}\mathrm{e}^{-it}[-\frac{iW_2}{k} + \frac{ig}{k} + iC_0 + iC_1 H_2 - \frac{C_1 R}{\Omega \cdot AS} - C_1\frac{m}{\Omega}W_2 + \frac{\dot{W}_2}{k}]. \tag{14.6}$$

The derivative of $\mathcal{H}_2$ with respect to the water table elevation $H_2$ is given by

$$-\frac{\partial \mathcal{H}_2}{\partial H_2} = -C_1 W_2 \mathrm{e}^{-it}. \quad (14.7)$$

From Equation (14.3) and (14.6), we obtain the following equation.

$$-C_1 W_2 = \frac{\Omega}{m}[-\frac{iW_2}{k} + \frac{ig}{k} + iC_0 + iC_1 H_2 - \frac{C_1 R}{\Omega \cdot AS} - C_1 \frac{m}{\Omega} W_2$$

$$+\frac{\dot{W}_2}{k}]. \quad (14.8)$$

Solving for $\dot{W}_2$ in the above equation we get the following equations.

$$\frac{\dot{\Omega} \cdot W_2}{mk} = \frac{\Omega \cdot iW_2}{mk} - \frac{\Omega \cdot ig}{mk} - \frac{\Omega \cdot iC_0}{m} - \frac{\Omega \cdot iC_1 H_2}{m} + \frac{\Omega \cdot C_1 R}{m\Omega \cdot AS}$$

$$+\frac{\Omega \cdot C_1 m W_2}{\Omega \cdot m} - C_1 W_2 \quad (14.9)$$

$$\frac{\dot{W}_2}{k} = \frac{iW_2}{k} - \frac{ig}{k} - iC_0 - iC_1 H_2 + \frac{C_1 R}{\Omega AS}$$

$$+C_1 m W_2 - C_1 m W_2 \quad (14.10)$$

$$\dot{W}_2 = iW_2 - ig - iC_0 k - iC_1 H_2 k + \frac{kC_1 R}{\Omega \cdot AS} \quad (14.11)$$

$$\dot{W}_2 = iW_2 - ikC_1 H_2 + [-ig - ikC_0 + \frac{C_1 kR}{\Omega \cdot AS}]. \quad (14.12)$$

Likewise, the value for $\dot{H}_2$ can be rewritten as

$$\dot{H}_2 = \frac{(\alpha-1)W_2}{\Omega \cdot AS} + \frac{R}{\Omega \cdot AS}. \quad (14.13)$$

Consequently, we now have to solve the two simultaneous differential equations ((14.12) and (14.13)). Thus, by letting $mm = \frac{(\alpha-1)}{\Omega \cdot AS}$, $u = ikC_1$, $NN1 = -ig - ikC_0 + \frac{C_1 kR}{\Omega \cdot AS}$ and $MM = \frac{R}{\Omega \cdot AS}$, we get the following system of differential equations.

$$\dot{W}_2 = iW_2 - u \cdot H_2 + NN1. \quad (14.14)$$

$$\dot{H}_2 = mm \cdot W_2 + MM. \quad (14.15)$$

Putting the above system of differential equations in a $D$ operator format (where $D = \frac{d}{dt}$), and solving for $W_2$ yields the following second order linear non-homogeneous differential equation.

$$[(D^2 - Di) + u \cdot mm]W_2 = -u \cdot MM. \tag{14.16}$$

The particular solution of the above differential equation is given by: $-\frac{MM}{mm}$ and the solution to the homogeneous differential equation ($[(D^2 - Di) + u \cdot mm]W_2 = 0$) by

$$W_2(t) = \overline{EA1}e^{tz_1} + \overline{EB1}e^{tz_2}, \tag{4.17}$$

where $z_{1,2} = \frac{i \pm \sqrt{i^2 - 4umm}}{2}$ are the characteristic roots. The parameters $\overline{EA1}$ and $\overline{EB1}$ are constants to be determined by imposing the initial conditions. Substituting the right hand side (RHS) of (4.17) for $W(t)$ in the homogenous DE ($\dot{H}_2 = mm \cdot W_2$) and integrating gives the solution for the water table level $H(t)$ as follows.

$$H_2(t) = \frac{mm \cdot \overline{EA1}}{z_1}e^{tz_1} + \frac{mm \cdot \overline{EB1}}{z_2}e^{tz_2}. \tag{14.18}$$

Furthermore, the steady state level water table is given by

$$H_2^* = [\frac{-i\frac{MM}{mm} + NN1}{u}] \tag{14.19}$$

Hence, the solution for $W_2^*(t)$ and $W_2^*(t)$ are given as follows, respectively.

$$W_2(t) = \overline{EA1}e^{tz_1} + \overline{EB1}e^{tz_2} - \frac{MM}{mm}, \tag{14.20}$$

$$H_2(t) = \frac{mm \cdot \overline{EA1}}{z_1}e^{tx_1} + \frac{mm \cdot \overline{EB1}}{z_2}e^{tx_2} + \frac{NN1 - i\frac{MM}{mm}}{u}. \tag{14.21}$$

Similarly to Gisser and Sanchez (1980) results, it is worth mentioning that $-4umm > 0$ since $k < 0$, $C_1 < 0$, $i > 0$, $A > 0$, $S > 0$, $\Omega > 0$, and $\alpha < 1 \Longrightarrow (\alpha - 1) < 0$. This implies that $z_1 > i$ and $z_2 < 0$. Therefore, $z_2$ is the stable characteristic root. Likewise, similarly to Gisser and Sanchez (1980), we obtained that the transversality condition is only satisfied when $\overline{EA1} = 0$. By imposing the initial conditions of the sub problem ($H_2(t_T) = H_T$), we obtain the constant $\overline{EB}$ as follows below.

$$\overline{EB1} = \frac{z_2}{mm}[H_T - \frac{NN1 - i\frac{MM}{mm}}{u}]e^{-z_2 t_T}. \tag{14.22}$$

Therefore, the optimal solutions for $W_2^*(t)$ and $H_2^*(t)$ are given as follows below, respectively.

$$W_2^*(t) = \frac{z_2}{mm}[H_T - \frac{NN1 - i\frac{MM}{mm}}{u}]e^{z_2(t - t_T)} - \frac{MM}{mm}. \tag{14.23}$$

$$H_2^*(t) = [H_T - \frac{NN1 - i\frac{MM}{mm}}{u}]e^{z_2(t - t_T)} + \frac{NN1 - i\frac{MM}{mm}}{u}. \tag{14.24}$$

We can now solve the first sub-problem since we have the solution ($SP_2^*$) to the second sub problem. The hamiltonian function of the system in the first phase is given as follows

$$\mathcal{H}_1(t, W_1, H_1, \lambda_2) = -\mathrm{e}^{-it}[\frac{W_1^2}{2k} - \frac{gW_1}{k} - (C_0 + C_1H_1)W_1]$$

$$+\lambda_1 \cdot \frac{[R+(\alpha-1)W_1]}{AS} \quad (14.25)$$

Hence, the first order conditions are as follows

$$\frac{\partial \mathcal{H}_1}{\partial W_1} = -\mathrm{e}^{-it}[\frac{W_1}{k} - \frac{g}{k} - C_0 - C_1H_1] + \lambda_1[\frac{(\alpha-1)}{AS}] = 0. \quad (14.26)$$

$$\dot{\lambda}_1 = -\frac{\partial \mathcal{H}_1}{\partial H_1}. \quad (14.27)$$

$$\lambda_1^*(t_T, W_1^*(t_T), H_1^*(t_T)) = \lambda_2^*(t_T, W_2^*(t_T), H_2^*(t_T)). \quad (14.28)$$

$$H_1^*(t_T, W_1^*(t_T), H_1^*(t_T)) = \frac{\partial SP_2^*(t_T, W_1^*(t_T), H_1^*(t_T))}{\partial t_T}. \quad (14.29)$$

$$\dot{H}_1 = \frac{1}{AS}[R + (\alpha - 1)W_1]. \quad (14.30)$$

The transversality condition is given by $\lim_{t\to\infty}\lambda_1(t) = 0$. From Equation (14.26), we obtain the value for the costate variable $\lambda_1$ as follows.

$$\lambda_1 = \frac{1}{m}\mathrm{e}^{-it}[(\frac{1}{k})W_1 - \frac{g}{k} - C_0 - C_1H_1], \quad (14.31)$$

where $m = \frac{(\alpha-1)}{AS}$. The derivative of $\lambda_1$ with respect to $t$ is given by

$$\dot{\lambda}_1 = \frac{1}{m}\mathrm{e}^{-it}[-\frac{iW_1}{k} + \frac{ig}{k} + iC_0 + iC_1H_1 - \frac{C_1R}{AS} - C_1mW_1 + \frac{\dot{W}_1}{k}]. \quad (14.32)$$

The derivative of $\mathcal{H}_1$ with respect to the water table elevation $H_1$ is given by

$$-\frac{\partial \mathcal{H}_1}{\partial H_1} = -C_1W_1\mathrm{e}^{-it}. \quad (14.33)$$

From Equation (14.27) and (14.32), we obtain the following equation.

$$-C_1W_1 = \frac{1}{m}[-\frac{iW_1}{k} + \frac{ig}{k} + iC_0 + iC_1H_1 - \frac{C_1R}{AS} - C_1mW_1$$

$$+\frac{\dot{W}_1}{k}]. \quad (14.34)$$

Solving for $\dot{W}_1$ in the above equation we get the following equations.

$$\frac{\dot{W}_1}{mk} = \frac{iW_1}{mk} - \frac{ig}{mk} - \frac{iC_0}{m} - \frac{iC_1H_1}{m} + \frac{C_1R}{mAS} + \frac{C_1mW_1}{m} - C_1W_1 \quad (14.35)$$

$$\frac{\dot{W}_1}{k} = \frac{iW_1}{k} - \frac{ig}{k} - iC_0 - iC_1H_1 + \frac{C_1R}{AS} + C_1mW_1 - C_1mW_1 \quad (14.36)$$

$$\dot{W}_1 = iW_1 - ig - iC_0k - iC_1H_1k + \frac{kC_1R}{AS} \quad (14.37)$$

$$\dot{W}_1 = iW_1 - ikC_1H_1 + [-ig - ikC_0 + \frac{C_1kR}{AS}]. \quad (14.38)$$

Likewise, the value for $\dot{H}_1$ can be rewritten as

$$\dot{H}_1 = \frac{(\alpha-1)W_1}{AS} + \frac{R}{AS}. \quad (14.39)$$

Consequently, we now have to solve the two simultaneous differential equations ((14.38) and (14.39)). Thus, by letting $m = \frac{(\alpha-1)}{AS}$, $u = ikC_1$, $N_0 = -ig - ikC_0 - ikG_1 + \frac{C_1kR}{AS}$ and $M = \frac{R}{AS}$, we get the following system of differential equations.

$$\dot{W}_1 = iW_1 - u \cdot H_1 + N_0. \quad (14.40)$$

$$\dot{H}_1 = m \cdot W_1 + M. \quad (14.41)$$

Putting the above system of differential equations in a $D$ operator format (where $D = \frac{d}{dt}$), and solving for $W_1$ yields the following second order linear non-homogeneous differential equation.

$$[(D^2 - Di) + u \cdot m]W_1 = -u \cdot M. \quad (14.42)$$

The particular solution of the above differential equation is given by: $-\frac{M}{m}$ and the characteristic roots by $y_{1,2} = \frac{i \pm \sqrt{i^2 - 4um}}{2}$. Furthermore, the steady state level water table is given by

$$H_1^* = [\frac{-i\frac{M}{m}+N_0}{u}] \tag{14.43}$$

Hence, the solution for $W_1^*(t)$ and $H_1^*(t)$ is given by

$$W_1^*(t) = \widetilde{A1}\mathrm{e}^{y_1 t} + \widetilde{B1}\mathrm{e}^{y_2 t} - \frac{M}{m}. \tag{14.44}$$

$$H_1^*(t) = \frac{m}{y_1}\widetilde{A1}\mathrm{e}^{y_1 t} + \frac{m}{y_2}\widetilde{B1}\mathrm{e}^{y_2 t} + \frac{N_0 - i\frac{M}{m}}{u}. \tag{14.45}$$

Where $\widetilde{A1}$ and $\widetilde{B1}$ are obtained by imposing the initial conditions.

$$\widetilde{B1} = \frac{y_2 AS}{\alpha-1}[H_0 - \frac{N_0-\frac{iR}{\alpha-1}}{ikC_1} - \frac{[H_T-\frac{N_0-\frac{iR}{\alpha-1}}{ikC_1}]-[H_0-\frac{N_0-\frac{iR}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2 t_T}}{\mathrm{e}^{y_1 t_T}-\mathrm{e}^{y_2 t_T}}], \tag{14.46}$$

$$\widetilde{A1} = \frac{y_1 AS}{\alpha-1}[\frac{[H_T-\frac{N_0-\frac{iR}{\alpha-1}}{ikC_1}]-[H_0-\frac{N_0-\frac{iR}{\alpha-1}}{ikC_1}]\mathrm{e}^{y_2 t_T}}{\mathrm{e}^{y_1 t_T}-\mathrm{e}^{y_2 t_T}}]. \tag{14.47}$$

The maximization principle specifies the necessary conditions for optimality. However, it is also necessary to ensure that the second-order conditions are met. The compliance of the second order conditions ensures that the maximum principle's necessary conditions are likewise sufficient for global optimality. Mangasarian established a basic sufficiency theorem (Chiang 1992, pp. 214–217) that guarantees the second order conditions. In this problem, the sufficient conditions of the Mangasarian theorem have been verified, allowing us to conclude that the obtained trajectories are optimal.

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